Combining Symmetry Operations (Multiplication)

- Multiplication of symmetry operations is the successive performance of two or more operations to achieve an orientation that could be reached by a single operation.

- The order in which successive different symmetry operations are performed can affect the result.

- Multiplication of symmetry operations is not in general commutative, although certain combinations may be.

- In writing multiplications of symmetry operation we use a "right-to-left" notation:

  \[ BA = X \]

  "Doing \( A \) then \( B \) has the same result as the operation \( X \)."

  - We cannot assume that reversing the order will have the same result.

  - It may be that either \( BA \neq AB \) or \( BA = AB \).

- Multiplication is associative:

  \[ C(BA) = (CB)A \]
Fig. 1.12 The order of performing $S_4$ and $\sigma_v$, shown here for a tetrahedral MX$_4$ molecule, affects the result. The final positions in each case are not the same, but they are related to each other by $C_2$. 
All possible binary combinations of symmetry operations can be summarized in a multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>C</td>
<td>E</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

Combination order is "top" then "side"; e.g.,

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>EE = E</td>
<td>EA = A</td>
<td>EB = B</td>
<td>EC = C</td>
</tr>
<tr>
<td>A</td>
<td>AE = A</td>
<td>AA = B</td>
<td>AB = C</td>
<td>AC = E</td>
</tr>
<tr>
<td>B</td>
<td>BE = B</td>
<td>BA = C</td>
<td>BB = E</td>
<td>BC = A</td>
</tr>
<tr>
<td>C</td>
<td>CE = C</td>
<td>CA = E</td>
<td>CB = A</td>
<td>CC = B</td>
</tr>
</tbody>
</table>
Fig. 1.13  Symmetry elements of CBr₂Cl₂.
Matrix Notation of the Effects of the Operations

\[
\begin{bmatrix}
E
\end{bmatrix} \times \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix} = \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_2
\end{bmatrix} \times \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix} = \begin{bmatrix}
Br_b \\
Br_a \\
Cl_b \\
Cl_a \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_v
\end{bmatrix} \times \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix} = \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma'_{v}
\end{bmatrix} \times \begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b \\
\end{bmatrix} = \begin{bmatrix}
Br_a \\
Br_b \\
Cl_b \\
Cl_a \\
\end{bmatrix}
\]
### Multiplication Table for the Operations of CBr$_2$Cl$_2$

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>C$_2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>C$_2$</td>
<td>C$_2$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$E$</td>
<td></td>
</tr>
</tbody>
</table>

**Step 1: Combinations with identity.**

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>C$_2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>C$_2$</td>
<td>C$_2$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$E$</td>
<td></td>
</tr>
</tbody>
</table>

**Step 2: Binary self-combinations.**

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>C$_2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>C$_2$</td>
<td>C$_2$</td>
<td>C$_2$</td>
<td>$\sigma_v$</td>
<td>$E$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$E$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$E$</td>
<td></td>
</tr>
</tbody>
</table>
Step 3: Mixed binary combinations.

\[ C_2\sigma_v = ? \]

\[
\begin{bmatrix}
\sigma_v'
\end{bmatrix} \times
\begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b
\end{bmatrix} =
\begin{bmatrix}
Br_a \\
Br_b \\
Cl_b \\
Cl_a
\end{bmatrix}
\]

\[
[C_2] \times
\begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b
\end{bmatrix} =
\begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b
\end{bmatrix}
\]

This result is the same as that achieved by \( \sigma_v' \) alone:

\[
\begin{bmatrix}
\sigma_v
\end{bmatrix} \times
\begin{bmatrix}
Br_a \\
Br_b \\
Cl_a \\
Cl_b
\end{bmatrix} =
\begin{bmatrix}
Br_b \\
Br_a \\
Cl_a \\
Cl_b
\end{bmatrix}
\]

\( C_2\sigma_v = \sigma_v' \)
Complete Multiplication Table

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$C_2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$E$</td>
<td>$C_2$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$C_2$</td>
<td>$E$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v$</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>$\sigma_v$</td>
<td>$\sigma_v'$</td>
<td>$E$</td>
<td>$C_2$</td>
</tr>
<tr>
<td>$\sigma_v'$</td>
<td>$\sigma_v'$</td>
<td>$\sigma_v$</td>
<td>$C_2$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

General Results:

- ✔ The first row of results duplicates the list of operations in the header row.
- ✔ The first column of results duplicates the list of operations in the label column.
- ✔ Every row shows every operation once and only once.
- ✔ Every column shows every operation once and only once.
- ✔ The order of resultant operations in every row is different from any other row.
- ✔ The order of resultant operations in every column is different from any other column.